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Line-density and Hamiltonian density, Random Bunch Set-up

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Aim: We assume an equilibrium bunch in a synchrotron with multi-frequency RF system and possibly energy loss or acceleration per turn; for this bunch a linear line-density $\lambda(\tau)$ is given. We want first to determine a mathematical expression for the density function ρ in 2-dimensional longitudinal phase space having this τ-projection $\lambda(\tau)$ and then a randomized set of macro-particles corresponding to it.

Conventions:
- The longitudinal position of a particle in the bunch is expressed by their arrival time τ, hence particles with larger τ arrive later.
- The τ-location where there is no acceleration, i.e. $eV_{RF}(\tau) - \Delta U = 0$, ΔU expressing the sum of energy loss and acceleration per machine turn, is called τ_0 (equivalent to the synchronous phase).
- We assume an equilibrium bunch, i.e. particles leaving a static phase-space volume element are always replaced by the same number of particles entering it.
- For such an equilibrium bunch the projection $\lambda(\tau)$ for $\tau \geq \tau_0$ determines the *complete* phase space population and hence also $\lambda(\tau)$ for $\tau < \tau_0$ and vice-versa. Therefore we can only specify the line density for one side of τ_0, the other side is also determined then.
- During the synchrotron oscillation particles move between their most positive τ to their most negative τ-excursion, creating a contribution to $\lambda(\tau)$ everywhere in-between. For a realistic *non-negative* phase-space density the function $\lambda(\tau)$ can then never be less than the contribution from the outer population. Therefore not any arbitrary function $\lambda(\tau)$ corresponds to a real non-negative phase-space density.

The phase space dynamic

Each particle has a longitudinal position τ and a deviation from the nominal particle energy, U. On each turn the particle feels the *total* RF voltage $V_{RF}(\tau)$ and the constant energy change per turn $\Delta U = \Delta U_{loss} + \Delta U_{acc}$, the sum of a true energy loss ΔU_{loss} and a nominal energy increase due to acceleration ΔU_{acc}. Then the particle's total energy change per turn is

(1) $$U' = U + eV_{RF}(\tau) - \Delta U \quad \Rightarrow \quad \frac{dU}{dt} = \frac{eV_{RF}(\tau) - \Delta U}{T_{rev}}$$

The second expression – T_{rev} being the nominal revolution time – is the smooth differential *approximation.* With a constant α – function of the machine's transition gamma γ_t and the particle momentum p_0 – a particle with energy deviation U' drifts along the machine from arrival time τ to τ' or from longitudinal position τ to τ' at the next turn as

(2) $$\tau' = \tau + \alpha \cdot U' \Rightarrow \frac{d\tau}{dt} = \frac{\alpha}{T_{rev}} \cdot U;$$

Above transition energy a more energetic particle with U' > 0 arrives later than at the last turn, i.e. τ' > τ, equivalent $\alpha > 0$. Again, the second expression is the smooth differential *approximation.*

The differential approximations in (1) and (2) represent a system with time-independent force[1] (conservative system) and can be derived from a Hamiltonian function $\overline{H}(U,z)$ provided it holds the two relations

(3) $$-\frac{\partial \overline{H}}{\partial \tau} = \frac{dU}{dt}; \qquad \frac{\partial \overline{H}}{\partial U} = \frac{d\tau}{dt}$$

We define the potential P(τ) – with an irrelevant integration constant – by[2]

(4) $$P(\tau) = -\int \left(eV_{RF}(\tau) - \Delta U\right)\, d\tau = \Delta U \cdot \tau - e\int V_{RF}(\tau)\, d\tau$$

and define

(5) $$\overline{H}(U,\tau) = \frac{\frac{1}{2}\alpha \cdot U^2 + P(\tau)}{T_{rev}}$$

for which we have in fact

(6) $$-\frac{\partial \overline{H}}{\partial \tau} = \frac{eV_{RF}(\tau) - \Delta U}{T_{rev}} \equiv \frac{dU}{dt}$$

(7) $$\frac{\partial \overline{H}}{\partial U} = \frac{\alpha}{T_{rev}} \cdot U \equiv \frac{d\tau}{dt}$$

hence (5) is Hamiltonian of this system. This means that in a phase-space defined for these canonically conjugate variables U and τ the local particle density

(8) $$\rho(\tau,U) = \frac{d^2 n}{d\tau\; dU}$$

does not change while any differential volume is displaced from turn to turn (Liouville theorem). Since we consider an equilibrium bunch where particles leaving a *static* phase-space volume are exactly replaced by new ones arriving, this also means that ρ is constant for all coordinates (U,τ) that correspond to the same Hamiltonian. To simplify for future use, we multiply (5) by T_{rev}, yielding an easier function H

(9) $$H = \tfrac{1}{2}\alpha \cdot U^2 + P(\tau)$$

that is also constant when (5) is constant.

Determination of the phase space density F(H) from λ(τ)

For any longitudinal phase space distribution in accelerators ρ is identical for –U and U. L(τ) may describe the number of particles that have a τ-coordinate below τ, hence L is a <u>scalar function</u>, i.e. its value is independent under variable transformation of τ. Considering only one bucket with the lower end at $\tau=\tau_{min}$ we get

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[1] The RF voltage is 'probed' only stroboscopically at times being integer multiple (harmonic number) of the RF oscillation time, hence in this context the effective RF voltage appears independent of time, a function of τ only.

[2] above transition energy V_{RF} around the synchronous time has to fall with increasing time. Then P(τ) as defined in (4) has a *minimum* around the synchronous location τ_0 as it should be. Below transition energy P would have a maximum and α has to become less than zero for a stable bunch; then H → -H reestablishes the usual situation with a minimum for –P(τ).

(10) $$L(\tau) = \int_{\tau_{min}}^{\tau} \int_{-\infty}^{\infty} \rho(\tau,U)\; dU \;\; d\tau = 2 \int_{\tau_{min}}^{\tau} \int_{0}^{\infty} \rho(\tau,U)\; dU \;\; d\tau$$

We call $\lambda_\tau(\tau)$ the directly measurable longitudinal τ-projected density between τ and τ+dτ

(11) $$\lambda_\tau(\tau) = \frac{dL(\tau)}{d\tau} = 2 \int_{0}^{\infty} \rho(\tau,U)\; dU$$

In contrast to L(τ) the *density function* $\lambda_\tau(\tau)$ is not independent under transformation of τ to another variable; therefore we prefer to use the scalar L(τ) in the following calculations.

We transform the double integral (10) to the variables P and H. P(τ) is the potential function, monotonous in τ in the considered range on one side of τ_0 only, else there would be an expelling force and no bound particle movement. Then there is a one-to-one correspondence between τ and P, i.e. P(τ) can be inverted.

The differential transformation matrix from τ and U to P and H of (9) becomes

(12) $$\begin{pmatrix} dH \\ dP \end{pmatrix} = \begin{pmatrix} \alpha\cdot U & \frac{dP}{d\tau}(p) \\ 0 & \frac{dP}{d\tau}(p) \end{pmatrix} \cdot \begin{pmatrix} dU \\ d\tau \end{pmatrix} \Rightarrow \left|\alpha\cdot U\cdot \frac{dP}{d\tau}\right| dU\cdot d\tau = dH\cdot dP$$

hence with $2(H-P)/\alpha = U^2$

(13) $$dU\cdot d\tau = \sqrt{\frac{1}{2\alpha}} \frac{d\tau}{dP}(P)\cdot \frac{dH\cdot dP}{\sqrt{H-P}}$$

The number of particles with a P-coordinate below P=P(τ) then becomes

(14) $$\bar{L}(P) = \sqrt{\frac{2}{\alpha}} \int_{P_{min}}^{P} d\bar{P} \int_{\bar{P}}^{\infty} \frac{\rho\left(\tau(\bar{P}),U(H,\bar{P})\right)}{\sqrt{H-\bar{P}}} \frac{d\tau}{dP}(\bar{P})\; dH$$

ρ has the same value for the same H, it is a function of H only: $\rho(\tau,U)$ = F(H). Remind that F is a *density* expressed for the phase space *volume element* dτ·dU !!

(15) $$\bar{L}(P) = \sqrt{\frac{2}{\alpha}} \int_{P_{min}}^{P} d\bar{P} \int_{\bar{P}}^{\infty} \frac{F(H)}{\sqrt{H-\bar{P}}} \frac{d\tau}{dP}(\bar{P})\; dH$$

Differentiating this expression with respect to τ yields the density function λ_τ (expressing density per step-width dτ, not dP!) but expressed for the argument P – P being uniquely related to a corresponding τ

(16) $$\lambda_\tau(P) = \frac{d\bar{L}}{dP}\frac{dP}{d\tau} = \sqrt{\frac{2}{\alpha}} \frac{dP}{d\tau}\cdot \int_{\bar{P}}^{\infty} \frac{F(H)}{\sqrt{H-\bar{P}}} \frac{d\tau}{dP}(\bar{P})\; dH$$

Since we integrate over H while P is independent variable, we can pull the factor dτ/dP(P), independent of H, out of the integral, cancelling the factor dP/dτ

(17) $$\lambda_\tau(P) = \sqrt{\frac{2}{\alpha}}\cdot \int_{\bar{P}}^{\infty} \frac{F(H)}{\sqrt{H-\bar{P}}}\; dH$$

We define the function G(P) as

(18) $$G(P) = \sqrt{\alpha/2}\cdot \lambda_\tau(P) = \int_{P}^{\infty} \frac{F(H)}{\sqrt{H-P}}\; dH$$

We can use the mathematical equivalence (proof in Appendix 1)

(19) $$G(P) = \int_P^\infty \frac{F(H)}{\sqrt{H-P}}\,dH \quad \Leftrightarrow \quad F(H) = -\frac{1}{\pi}\int_H^\infty \frac{dG}{dP}\frac{1}{\sqrt{P-H}}\,dP$$

to conclude

(20) $$\boxed{F(H) = -\frac{1}{\pi}\sqrt{\frac{\alpha}{2}}\cdot\int_H^\infty \frac{dP}{\sqrt{P-H}}\frac{d\lambda_\tau}{dP}}$$

The initial $\lambda_\tau(\tau)=dL/d\tau$ can be expressed as function of P, i.e. $\lambda_\tau(P)$ while keeping the density per $d\tau$, by the unique mapping $\tau=P^{-1}(p)$ which then allows to calculate (20). We will come back to this method in the numerical part again.

To express F(H) as integral over τ without use of the inverse function P^{-1} we define $\hat{\tau}$ as the coordinate where the ring of constant H passes the line U=0, i.e. $H = P(\hat{\tau})$ and with it $\hat{\tau} = P^{-1}(H)$ (using P^{-1} only for a single point). Then applying

(21) $$\frac{d\lambda_\tau}{dP}\,dP = \frac{d\lambda_\tau}{d\tau}\frac{d\tau}{dP}\,dP = \frac{d\lambda_\tau}{d\tau}\,d\tau$$

yields the integration by τ

(22a) $$\boxed{F(H) = -\frac{1}{\pi}\sqrt{\frac{\alpha}{2}}\cdot\int_{\hat{\tau}}^\infty \frac{d\tau}{\sqrt{P(\tau)-H}}\frac{d\lambda_\tau}{d\tau}(\tau)}$$

or expressing F directly as function of $\hat{\tau}$ – avoiding any inversion of P – and correspondingly replacing $H = P(\hat{\tau})$ yields

(22b) $$\boxed{F(\hat{\tau}) = -\frac{1}{\pi}\sqrt{\frac{\alpha}{2}}\cdot\int_{\hat{\tau}}^\infty \frac{d\tau}{\sqrt{P(\tau)-P(\hat{\tau})}}\frac{d\lambda_\tau}{d\tau}(\tau)}$$

In all cases $F(H)=\rho(\tau,U)$ is the 2D phase space density function per volume element $d\tau\cdot dU$, λ_τ the 1D projection density per $d\tau$.

If λ_τ is a function clipped to zero outside $\tau=\tau_{\text{lim}}$ while $\lambda_\tau(\tau_{\text{lim}}^-) > 0$, the derivative $d\lambda_\tau/d\tau$ is a delta-function and the additional contribution ΔF with

(23a) $$\Delta F(H) = +\frac{1}{\pi}\sqrt{\frac{\alpha}{2}}\frac{\lambda_\tau(\tau_{\text{lim}}^-)}{\sqrt{P(\tau_{\text{lim}})-H}}$$ equivalent

(23b) $$\Delta F(\hat{\tau}) = +\frac{1}{\pi}\sqrt{\frac{\alpha}{2}}\frac{\lambda_\tau(\tau_{\text{lim}}^-)}{\sqrt{P(\tau_{\text{lim}})-P(\hat{\tau})}}$$

has to be added to the 'regular' term. (23) remains finite as long as $\hat{\tau}$ does not reach τ_{lim} (equivalent H does not reach $P(\tau_{\text{lim}})$), but there it will diverge if $\lambda(\tau_{\text{lim}}^-) > 0$. This means that for an infinitely steep rise (step) of the true line density F will diverge; but the particle count inside a ring between $\hat{\tau}$ and $\hat{\tau}+d\hat{\tau}$ (H and H+dH), proportional to the integral over F from $\hat{\tau}$ and $\hat{\tau}+d\hat{\tau}$ (H to H+dH), remains finite.

Especially this means that the so much liked Gaussian bunch profile, that extends mathematically to infinity and hence has to be clipped somewhere with $\lambda_\tau(\tau_{\text{lim}}^-) > 0$ to stay within the bucket, causes some problems or special arrangements have to be made.

Examples for transformations between F(H) and λ(τ)

Example 1:
Let $\lambda_\tau(\tau)$ have a *rectangular* shape

(24) $$\lambda_\tau(\tau) = \begin{cases} \lambda_0 & |\tau| \le \tau_{max} \\ 0 & else \end{cases} \quad \text{and hence} \quad \frac{d\lambda_\tau}{d\tau} = \begin{cases} -\delta(\tau)\cdot\lambda_0 & |\tau| = \tau_{max} \\ 0 & else \end{cases}$$

In this case the integral (22) over the constant-λ part is zero due to $d\lambda_\tau/d\tau=0$, only the border contribution appears as the delta-integral as shown in (23) using $H_{max}=P(\tau_{max})$

(25) $$F(H) = \frac{\lambda_0}{\pi}\sqrt{\frac{\alpha}{2}}\cdot\frac{1}{\sqrt{H_{max} - H}}$$

The density function F(H) itself diverges as $H \to H_{max}$ but the integral over it, the particles count, remains finite (the integral over $1/\sqrt{x}$ remains finite as $x \to 0$).
For this F(H) we now back-calculate $\lambda_\tau(\tau)$ and get for $\tau \le \tau_{max}$ with the U-independent constant $A^2 = 2(H_{max} - P(\tau))/\alpha$

(26) $$\lambda_\tau(\tau) = \int_{-\infty}^{+\infty} F\left(\tfrac{1}{2}\alpha U^2 + P(\tau)\right)\, dU = \frac{\lambda_0}{\pi}\int_{-A}^{+A}\cdot\frac{1}{\sqrt{A^2 - U^2}}\, dU$$

In fact we find back the initial line density

(27) $$\lambda_\tau(\tau) = \frac{\lambda_0}{\pi}\arcsin(U/A)\Big|_{-A}^{+A} \equiv \lambda_0$$

Example 2:
We assume that – unknown to us – the density F(H) is constant below the limiting H_{max}, else equal to zero (water-bag)

(28) $$F(H) = \begin{cases} \rho_0 & 0 \le H \le H_{max} \\ 0 & H_{max} < H \end{cases}$$

Then we will 'measure' as line density

(29) $$\lambda_\tau(\tau) = \int_{-\infty}^{+\infty} F\left(\tfrac{1}{2}\alpha U^2 + P(\tau)\right)\, dU = \int_{-\sqrt{2(H_{max}-P(\tau))/\alpha}}^{+\sqrt{2(H_{max}-P(\tau))/\alpha}} \rho_0\, dU = 2\rho_0\sqrt{\frac{2}{\alpha}}\,\sqrt{H_{max} - P(\tau)}$$

Now we assume that we do not know F(H) but only the line density (29) and try to figure out F(H). From (29) we get

(30) $$\frac{d\lambda_\tau}{d\tau} = -\sqrt{\frac{2}{\alpha}}\frac{\rho_0}{\sqrt{H_{max} - P(\tau)}}\frac{dP}{d\tau}$$

and thus using (22b)

(31) $$F(\hat{\tau}) = \frac{\rho_0}{\pi}\cdot\int_{\hat{\tau}}^{\tau_{max}} \frac{d\tau}{\sqrt{(P(\tau) - P(\hat{\tau}))(H_{max} - P(\tau))}}\frac{dP}{d\tau}$$

Replacing τ as integration variable by P and using the definitions $H = P(\hat{\tau})$ and $H_{max}=P(\tau_{max})$ yields

(32) $$F(H) = \frac{\rho_0}{\pi}\cdot\int_{H}^{H_{max}} \frac{dP}{\sqrt{(P - H)(H_{max} - P)}}$$

This integral is equal to

(33) $$F(H) = \frac{\rho_0}{\pi} \cdot \arctan\left(\frac{2P - H - H_{max}}{2 \cdot \sqrt{(P - H)(H_{max} - P)}} \right) \Bigg|_H^{H_{max}}$$

(34) $$F(H) = \frac{\rho_0}{\pi}(\arctan(+\infty) - \arctan(-\infty)) = \rho_0$$

i.e. we find F as put into the problem, as it should be.

Example 3:

From the previous example we can also deduce the shape of λ_τ for a single *differential* phase space ring at $H=H_0$

(35) $$F(H) = \hat{\rho}_0 \ \delta(H_0)$$

From (29) we know the line density λ_τ for constant $F=\rho_0$ between 0 and a given H_0. In calculating the difference of λ_τ for H_0+dH and for H_0, dividing by dH and form the limit for dH → 0 we get the desired result. The described procedure is the definition of the derivative, hence

(36) $$\lambda_\tau(\tau) = 2\hat{\rho}_0\sqrt{\frac{2}{\alpha}} \frac{d}{dH_0}\sqrt{H_0 - P(\tau)} = \sqrt{\frac{2}{\alpha}} \frac{\hat{\rho}_0}{\sqrt{H_0 - P(\tau)}}$$

We will not explicitly show the inversion here but rely on the calculation in the previous example.

Example 4:

Except for very special cases as above, to have examples that can be integrated explicitly, the potential function $P(\tau)$ has to allow such treatment; unfortunately for general $\lambda_\tau(\tau)$ this also excludes RF sine waves as $\sin(\tau\cdot\omega)$. Therefore we assume here a quadratic potential function (*linear oscillator, short bunches*) for the following example, i.e.

(37) $$P(\tau) = \tfrac{1}{2}k\tau^2$$

and thus

(38) $$H(U,\tau) = \tfrac{1}{2}\alpha U^2 + \tfrac{1}{2}k\tau^2$$

Now $\lambda(\tau)$ is assumed a parabolic line density (as sometimes assumed for proton bunches)

(39) $$\lambda_\tau(\tau) = \lambda_0\left(1 - (\tau/\tau_{max})^2\right); \quad \tau_{max} \geq \tau \geq \tau_0 = 0$$

and hence

(40) $$\frac{d\lambda_\tau}{d\tau} = -2\lambda_0/\tau_{max}^2 \cdot \tau$$

Then from (22a) and replacing $P(\hat{\tau}) = \tfrac{1}{2}k\hat{\tau}^2 = H$

(41) $$F(H) = \frac{\lambda_0\sqrt{2\alpha}}{\pi\ \tau_{max}^2} \cdot \int_{\hat{\tau}}^{\tau_{max}} \frac{\tau\ d\tau}{\sqrt{\tfrac{1}{2}k\tau^2 - H}} = \frac{\lambda_0\sqrt{2\alpha}}{\pi\ \tau_{max}^2 \cdot k} \cdot \sqrt{\tfrac{1}{2}k\tau_{max}^2 - H}$$

From this phase-space distribution we can calculate back the line density as

(42) $$\lambda_\tau(\tau) = \int_{-\infty}^{+\infty} F\left(\tfrac{1}{2}\alpha U^2 + \tfrac{1}{2}k\tau^2 \right) dU = \frac{\lambda_0 \cdot \alpha}{\pi\ \tau_{max}^2 \cdot k} \int_{-A}^{+A} \sqrt{A^2 - U^2}\, dU; \quad A^2 = k/\alpha \cdot \left(\tau_{max}^2 - \hat{\tau}^2\right)$$

with the U-independent constant A^2. As it should be, we get

(43) $$\lambda_\tau(\tau) = \frac{\lambda_0 \cdot \alpha \cdot A^2}{\pi\ \tau_{max}^2 \cdot k} \arcsin(U/A)\Big|_{-A}^{+A} = \lambda_0\left[1-\left(\tau/\tau_{max}\right)^2\right]$$

Numerical calculation of F(H) for general λ_τ and P(τ)

Now we want to determine the phase space density F(H) for an arbitrary potential P(τ) and line density (bunch profile) $\lambda_\tau(\tau)$. As already stated above, only functions λ_τ with certain properties will lead to a realistic non-negative F(H); to avoid problems we have to check that the resulting F(H) is really non-negative. Furthermore, the function λ_τ defined either for $\tau \geq \tau_0$ or for $\tau \leq \tau_0$ defines the whole phase space density F(H), hence also the complementary part of λ_τ. Therefore only one part has to be defined.
For arbitrary functions a closed solution for F(H) does not exist, apart rare exceptions, so we have to use numerical methods. The required integrals cannot be handled immediately by classical numerical integration methods since at the end of range at P→H the integrand diverges while the integral itself still remains finite.
In the integrals (22) around τ_0 the difference P(τ)-H becomes (about) proportional τ^2, hence a 1/τ term appears in the integral, which has to be counterbalanced by a purely linear τ-term in $d\lambda_\tau/d\tau$, i.e. $\lambda_\tau(\tau)$ has to be presented proportional to τ^2 with zero constant and linear term. To avoid this nuisance, it is easier to use integral (20) where the root of P-H creates a $1/\sqrt{P}$ term that remains finite during integration even with a constant numerator. The only slight difficulty is that the function P(τ) has to be numerically inverted; but this has to be done only once and can then be used for any integral.

Appendix 1

Let G(P) be defined by $G(P) = \int_P^\infty \frac{F(H)}{\sqrt{H-P}}\,dH$.

<u>Conjecture</u>: For $K(H) = -\frac{1}{\pi}\int_H^\infty \frac{dG}{dP}\frac{1}{\sqrt{P-H}}\,dP \quad \Leftrightarrow \quad K(H) \equiv F(H)$

<u>Proof</u>: By shift of variables we transform both integrals to have a *constant* lower bound

$$K(H) = -\frac{1}{\pi}\int_0^\infty \frac{dG}{dP}(y+H)\frac{dy}{\sqrt{y}}$$

$$G(P) = \int_0^\infty \frac{F(x+P)}{\sqrt{x}}\,dx \quad \Leftrightarrow \quad \frac{dG}{dP}(P) = \int_0^\infty F'(x+P)\,\frac{dx}{\sqrt{x}}$$

The latter derivative can be injected into the previous expression

$$K(H) = -\frac{1}{\pi}\int_0^\infty\int_0^\infty F'(x+y+H)\frac{dx\;dy}{\sqrt{x\cdot y}}$$

New variables ρ and ϕ are introduced[3]

$$x = \frac{\rho}{2}(1+\sin(\phi)) \quad \text{and} \quad y = \frac{\rho}{2}(1-\sin(\phi)).$$

These variables hold the relations x+y=ρ and x-y=ρ·sin(ϕ). The range $0 \le \rho < \infty$ and $-\pi/2 \le \phi \le +\pi/2$ covers exactly the initial range $0 \le x < \infty$ and $0 \le y < \infty$. It follows

$$\sqrt{x\cdot y} = \frac{\rho}{2}\sqrt{1-\sin^2(\phi)} = \frac{\rho\cdot\cos(\phi)}{2} \quad \text{and}$$

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{1+\sin(\phi)}{2} & \frac{\rho\cdot\cos(\phi)}{2} \\ \frac{1-\sin(\phi)}{2} & -\frac{\rho\cdot\cos(\phi)}{2} \end{pmatrix} \circ \begin{pmatrix} d\rho \\ d\phi \end{pmatrix}$$

The absolute value of the determinant of this matrix determines the transformation of the differential volume element as

$$dx\cdot dy = d\rho\cdot d\phi\cdot\frac{\rho\cdot\cos(\phi)}{2} \quad \text{hence}$$

$$K(H) = -\frac{1}{\pi}\int_0^\infty d\rho\int_{-\pi/2}^{+\pi/2} d\phi\;\; F'(\rho+H) = -\int_0^\infty F'(\rho+H)\;d\rho = F(H) - F(\infty)$$

The existence of the integral defining G(P) (first line) requires that F vanish for infinitely large arguments, hence

$$K(H) \equiv F(H) \qquad \text{q.e.d.}$$

Reading all these transformations in inverse order shows that the inverse conclusion also holds, i.e. for a F(H) as given in the conjecture the creating G(P) can be calculated as in the first line.

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[3] Motivation: • make x+y=ρ a single variable in F, x-y being the linear independent complementary variable • introduce polar-like coordinates so that the radial dependence of the root-term and the differential volume transformation term cancel • no angular dependence of x+y=ρ and for x-y=ρ·g(ϕ) an angular function g with simple result: For $g'^2 + g^2 = 1$ the transformation becomes unique with solution cos(ϕ) or sin(ϕ) (easier for boundaries)